# Algebras in representations of the symmetric group $S_t$, when $t$ is transcendental

Luke Sciarappa
Mentor: Nathan Harman

5th Annual PRIMES Conference
May 16, 2015

# Symmetry: groups, actions, representations

# Symmetry: groups, actions, representations

- Group: associative composition with identity and inverses

# Symmetry: groups, actions, representations

- Group: associative composition with identity and inverses

Examples

# Symmetry: groups, actions, representations

- Group: associative composition with identity and inverses

**Examples**

- Symmetries of a cube

# Symmetry: groups, actions, representations

- Group: associative composition with identity and inverses

Examples

- Symmetries of a cube
- $S_n$: ways to shuffle a pack of $n$ cards / permutations of $\{1, \ldots, n\}$

# Symmetry: groups, actions, representations

- Group: associative composition with identity and inverses

**Examples**

- Symmetries of a cube
- $S_n$: ways to shuffle a pack of $n$ cards / permutations of $\{1, \ldots, n\}$

- Action: group elements send objects to other objects

# Symmetry: groups, actions, representations

- Group: associative composition with identity and inverses

Examples

- Symmetries of a cube
- $S_n$: ways to shuffle a pack of $n$ cards / permutations of $\{1, \ldots, n\}$

- Action: group elements send objects to other objects

Examples

# Symmetry: groups, actions, representations

- Group: associative composition with identity and inverses

**Examples**

- Symmetries of a cube
- $S_n$: ways to shuffle a pack of $n$ cards / permutations of $\{1, \ldots, n\}$

- Action: group elements send objects to other objects

**Examples**

- Symmetries of a cube act on vertices/edges/faces of cube, and on $\mathbb{R}^3$

# Symmetry: groups, actions, representations

- Group: associative composition with identity and inverses

**Examples**

- Symmetries of a cube
- $S_n$: ways to shuffle a pack of $n$ cards / permutations of $\{1, \ldots, n\}$

- Action: group elements send objects to other objects

**Examples**

- Symmetries of a cube act on vertices/edges/faces of cube, and on $\mathbb{R}^3$

- Representation: group acting linearly on a vector space

# Categories

# Categories

- Category: mathematical objects; good notion of 'the right maps'

# Categories

- Category: mathematical objects; good notion of 'the right maps'

Examples

# Categories

- Category: mathematical objects; good notion of ‘the right maps’

**Examples**

- sets; functions

# Categories

- Category: mathematical objects; good notion of ‘the right maps’

**Examples**

- sets; functions
- vector spaces; linear maps

# Categories

- Category: mathematical objects; good notion of 'the right maps'

**Examples**

- sets; functions
- vector spaces; linear maps
- $G$-representations; $G$-linear maps (linear maps $\varphi$ s.t. $\varphi(g \cdot \mathbf{v}) = g \cdot \varphi(\mathbf{v})$)

# Tensor categories

# Tensor categories

- Tensor category: category w/lots of extra structure

# Tensor categories

- Tensor category: category w/lots of extra structure
- incl. tensor product: $(X, Y) \mapsto X \otimes Y$, associative, commutative, unital (ACU)

# Tensor categories

- Tensor category: category w/lots of extra structure
- incl. tensor product: $(X, Y) \mapsto X \otimes Y$, associative, commutative, unital (ACU)

Examples

# Tensor categories

- Tensor category: category w/lots of extra structure
- incl. tensor product: $(X, Y) \mapsto X \otimes Y$, associative, commutative, unital (ACU)

Examples

- Vector spaces

# Tensor categories

- Tensor category: category w/lots of extra structure
- incl. tensor product: $(X, Y) \mapsto X \otimes Y$, associative, commutative, unital (ACU)

**Examples**

- Vector spaces
- Representations of a finite group

# Tensor categories

- Tensor category: category w/lots of extra structure
- incl. tensor product: $(X, Y) \mapsto X \otimes Y$, associative, commutative, unital (ACU)

Examples

- Vector spaces
- Representations of a finite group
- In particular, for $n \in \mathbb{N}$, representations of $S_n =: \mathbf{Rep}(S_n)$

# Tensor categories

- Tensor category: category w/lots of extra structure
- incl. tensor product: $(X, Y) \mapsto X \otimes Y$, associative, commutative, unital (ACU)

Examples

- Vector spaces
- Representations of a finite group
- In particular, for $n \in \mathbb{N}$, representations of $S_n =: \mathbf{Rep}(S_n)$
- For $t \in \mathbb{C}$, $\mathrm{Rep}(S_t)$ ('interpolation' of $\mathbf{Rep}(S_n)$)

# Tensor categories

- Tensor category: category w/lots of extra structure
- incl. tensor product: $(X, Y) \mapsto X \otimes Y$, associative, commutative, unital (ACU)

Examples

- Vector spaces
- Representations of a finite group
- In particular, for $n \in \mathbb{N}$, representations of $S_n =: \mathbf{Rep}(S_n)$
- For $t \in \mathbb{C}$, $\mathrm{Rep}(S_t)$ ('interpolation' of $\mathbf{Rep}(S_n)$)

- $\mathrm{Rep}(S_t)$ early example of tensor category besides reps of a group

# Simple algebras

# Simple algebras

- Tensor category is good setting for commutative algebra

# Simple algebras

- Tensor category is good setting for commutative algebra
- Algebra $A$ has $m : A \otimes A \to A$ and $e : I \to A$; ACU

# Simple algebras

- Tensor category is good setting for commutative algebra
- Algebra $A$ has $m : A \otimes A \to A$ and $e : I \to A$; ACU
- Basic question of commutative algebra: what are simples?

# Simple algebras

- Tensor category is good setting for commutative algebra
- Algebra $A$ has $m : A \otimes A \to A$ and $e : I \to A$; ACU
- Basic question of commutative algebra: what are simples?
- Simple algebra: algebra with no nontrivial quotients

# Simple algebras

- Tensor category is good setting for commutative algebra
- Algebra $A$ has $m : A \otimes A \to A$ and $e : I \to A$; ACU
- Basic question of commutative algebra: what are simples?
- Simple algebra: algebra with no nontrivial quotients

**Examples**

# Simple algebras

- Tensor category is good setting for commutative algebra
- Algebra $A$ has $m : A \otimes A \to A$ and $e : I \to A$; ACU
- Basic question of commutative algebra: what are simples?
- Simple algebra: algebra with no nontrivial quotients

**Examples**

- Vector spaces — unique simple, $\mathbb{C}$

# Simple algebras

- Tensor category is good setting for commutative algebra
- Algebra $A$ has $m : A \otimes A \to A$ and $e : I \to A$; ACU
- Basic question of commutative algebra: what are simples?
- Simple algebra: algebra with no nontrivial quotients

**Examples**

- Vector spaces — unique simple, $\mathbb{C}$
- Representations of a finite group — simples come from subgroups

# Simple algebras

- Tensor category is good setting for commutative algebra
- Algebra $A$ has $m : A \otimes A \to A$ and $e : I \to A$; ACU
- Basic question of commutative algebra: what are simples?
- Simple algebra: algebra with no nontrivial quotients

**Examples**

- Vector spaces — unique simple, $\mathbb{C}$
- Representations of a finite group — simples come from subgroups
- Rep($S_t$) ('interpolation' of $\mathbf{Rep}(S_n)$) — ???

# Rep($S_t$)

# Rep($S_t$)

- P. Deligne, 2004

# Rep($S_t$)

- P. Deligne, 2004
- Schur-Weyl duality between $S_n$ and partition algebras $\mathbb{C}P_*(n)$

# Rep($S_t$)

- P. Deligne, 2004
- Schur-Weyl duality between $S_n$ and partition algebras $\mathbb{C}P_*(n)$
- $\mathbb{C}P_*(t)$ defined for any $t \in \mathbb{C}$

# Rep($S_t$)

- P. Deligne, 2004
- Schur-Weyl duality between $S_n$ and partition algebras $\mathbb{C}P_*(n)$
- $\mathbb{C}P_*(t)$ defined for any $t \in \mathbb{C}$
- Rep($S_t$), though no $S_t$

# Simple Algebras in Rep($S_t$)

# Simple Algebras in Rep($S_t$)

- Induction functor $\mathbf{Rep}(S_k) \boxtimes \mathrm{Rep}(S_{t-k}) \to \mathrm{Rep}(S_t)$, interpolating ordinary induction

# Simple Algebras in $\mathrm{Rep}(S_t)$

- Induction functor $\mathbf{Rep}(S_k) \boxtimes \mathrm{Rep}(S_{t-k}) \to \mathrm{Rep}(S_t)$, interpolating ordinary induction
- Inducing simple algebra from $\mathbf{Rep}(S_k)$ and trivial algebra from $\mathrm{Rep}(S_{t-k})$ gives simple algebra in $\mathrm{Rep}(S_t)$

# Simple Algebras in $\text{Rep}(S_t)$

- Induction functor $\textbf{Rep}(S_k) \boxtimes \text{Rep}(S_{t-k}) \to \text{Rep}(S_t)$, interpolating ordinary induction
- Inducing simple algebra from $\textbf{Rep}(S_k)$ and trivial algebra from $\text{Rep}(S_{t-k})$ gives simple algebra in $\text{Rep}(S_t)$

**Theorem**

*If $t$ is transcendental, every simple algebra in $\text{Rep}(S_t)$ is induced in this way from a simple algebra in $\textbf{Rep}(S_k)$ for some $k$.*

# Future directions

# Future directions

- Possible exotic algebras in algebraic $t$

# Future directions

- Possible exotic algebras in algebraic $t$
- Noncommutative algebras, Lie algebras

# Future directions

- Possible exotic algebras in algebraic $t$
- Noncommutative algebras, Lie algebras
- $\mathrm{Rep}(GL_t)$?

# Acknowledgments

Thanks to my mentor, Nathan Harman;
to Prof. Pavel Etingof, who suggested and supervised the project;
to the PRIMES program, for facilitating this research and conference;
to Dr. Tanya Khovanova, for advice on writing and presenting mathematics;
and
to my parents, Pauline and Ken Sciarappa, for transport and support.

# References

Jonathan Comes and Victor Ostrik.
On blocks of Deligne's category Rep($S_t$).
*Advances in Math.*, 226:1331–1377, 2011.
arXiv:0910.5695.

Pierre Deligne.
La catégorie des représentations du groupe symétrique $S_t$, lorsque $t$ n'est pas un entier naturel.
In *Proceedings of the International Colloquium on Algebraic Groups and Homogenous Spaces*, January 2004.

Pavel Etingof.
Representation theory in complex rank, I.
*Transformation Groups*, 19(2):359–381, 2014.
arXiv:1401.6321.

Akhil Mathew.
Categories parametrized by schemes and representation theory in complex rank.
*J. of Algebra*, 381, 2013.
arXiv:1006.1381.